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**Evaluation of "Q"**  
**in an Electrically Small Antenna**  
**in Prolate Spheroidal Coordinates**

R. C. Adams  
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## ABSTRACT

This document describes the energy distribution within an electrically small antenna. Chu (1948) derived expressions for the “Q” of an omnidirectional antenna whose entire structure is enclosed within a sphere. Many authors have extended this technique. Because the sphere is not conformal to the structure of most linear polarized antennas, much energy is not accounted for. If the structure is surrounded by an ellipsoid of revolution, the fit is much better. Prolate spheroidal coordinates are expected to provide a much better description for a linearly polarized antenna. We derive expressions for the electromagnetic field and the Poynting vector for vertically and horizontally polarized antennas. From the expressions, we calculate the Q of an electrically small antenna. We can derive an analytic expression for the value of Q for the leading term. The leading term is a product of the inverse of the cube of the dimensionless wave number (product of the wave number and half the distance between foci of the ellipse) and a factor that depends only upon the shape. This result is reformulated so that as the shape changes, the volume and wave number remain constant.

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## INTRODUCTION

Since 1947, many papers have been published on the radiation  $Q$  of electrically small antennas. With the exception of the original work by Wheeler (1947), most of these papers involved antennas contained within a sphere. In two of these papers, Watt (1967) and Simpson (2004) used circuit concepts to investigate antennas with cylindrical symmetry. Collin and Rothschild (1964) included an infinite cylinder. This document derives a result for the radiation  $Q$  of electrically small antennas contained within a prolate spheroid to understand the effect of aspect ratio defined as the height to width.

## THE INTERPRETATION AND IMPORTANCE OF RADIATION $Q$

As in the quality factor for a circuit, the radiation  $Q$  of an antenna is defined as  $2\pi$  times the ratio of the maximum energy stored to the total energy radiated per cycle. Radiation  $Q$  implies that only radiated power is considered dissipated. The radiation  $Q$  of an antenna is an important parameter. The available impedance bandwidth, radiation efficiency, and power-handling capability are inversely proportional to radiation  $Q$ . Thus, knowledge of the minimum radiation  $Q$  for a given antenna is important for design. The impedance bandwidth limitation implied by radiation  $Q$  can be overcome to some extent. The efficiency and power-handling limitations, however, cannot be overcome.

Geyi, Jarmuszewskik, and Qi (2000) showed that a one-port antenna is equivalent to a lossy network. The Foster reactance theorem applies and  $Q \approx f/BW$  provided  $Q \gg 1$ , where  $f$  is the resonant frequency and  $BW$  is the 3-dB bandwidth of the impedance. This bandwidth assumes a single reactive element tuning the antenna. The bandwidth determined in this way is really the bandwidth-efficiency product. Efficiency can be decreased to increase bandwidth (Simpson, 2004). Multiple tuning can significantly increase the impedance bandwidth. DeSantis' (1977) theoretical results showed that physically realizable lossless circuits limited by Foster's Reactance Theorem increase the impedance bandwidth of an electrically small monopole by more than three orders of magnitude.

The bandwidth limitations for transmission can be overcome in other ways. One way is to change the coupling to the antenna (Wheeler, 1947). A recent version of this approach used at U.S. Navy low-frequency (LF) stations uses a transmission line and network to transform the series resonance of the antenna to parallel resonance fed by a low-impedance, solid-state amplifier. The amount of bandwidth that can be enhanced by this method is a function of the excess KVA available in the transmitter. Another method of bandwidth enhancement at the U.S. Navy's very low-frequency (VLF) station at Cutler, Maine, uses variable reactive energy storage elements (Simpson, 2004).

The relationship between the radiation  $Q$  and radiation efficiency,  $\eta$ , is given by

$$\eta = R_r / (R_r + R_l) = 1 / (1 + Q_r / Q_l), \quad (1)$$

where  $R_r$  is the radiation resistance,  $R_l$  is the loss resistance in the tuning element (and other losses),  $Q_r$  is the radiation  $Q$ , and  $Q_l$  is the  $Q$  of the tuning element including the other losses. Equation (1) asserts that a lower radiation  $Q$  implies greater radiation efficiency. A higher  $Q$  implies higher voltages and currents as well as higher reactive fields near the antenna. The higher reactive fields result in increased loss.

For a given antenna, the voltage-limited radiated power is directly related to the radiation  $Q$ . The maximum voltage on the antenna is proportional to the product of  $Q_r$  and the square root of the power radiated. For a given power, this voltage depends only on the radiation  $Q$ , not the loaded  $Q$ . Lower  $Q$  is desired.

## PREVIOUS RESULTS

Wheeler (1947) was the first to investigate the concept of the minimum radiation  $Q$  that can be obtained for an electrically small antenna. He introduced the term "radiation power factor" as the inverse of the radiation  $Q$ . Using circuit concepts for a capacitor and inductor acting as an antenna, he showed that the radiation power factor for either kind of antenna is somewhat greater than

$$Q_r^{-1} > 4\pi^2/3 \cdot \text{Vol}/\lambda^3, \quad (2)$$

where  $\text{Vol}$  is the volume of the cylinder containing the antenna and  $\lambda$  is the wavelength. The same circuit concepts can be extended to show that the voltage-limited radiated power of an electrically short, top-loaded monopole is proportional to the square of the area (Watt, 1967).

Chu (1948) was the first to derive analytic results using Maxwell's equations for omnidirectional, linearly polarized radiation. He obtained expressions for the minimum radiation  $Q$  that can be obtained for an antenna that fit within a sphere of a given radius. This result assumed no energy remained within the sphere. The fields outside the sphere were expanded in spherical wave functions. These modes were orthogonal. The stored energy and radiated power were calculated for each mode using the complex Poynting Theorem and a ladder circuit derived for each mode. Chu obtained results for transverse-magnetic (TM) and transverse-electric (TE) linearly polarized and circularly polarized modes. Harrington (1960) extended this theoretical result to include the maximum possible gain. Other authors have expanded on this approach and derived exact results (Fante, 1969; McLean, 1996; Grimes and Grimes, 1995; and Fante and Mayhan, 1968) and have used the same type of theory to examine the power-handling capability of electrically small antennas by minimizing the electric field on a sphere around the antenna.

The literature is still somewhat controversial on the  $Q$  of electrically small antennas, especially for multiport antennas (Grimes and Grimes, 2001). For electrically small single-port antennas contained within a sphere, the results agree. When the sphere radius is less than one-tenth of a wavelength, all results reduce to the minimum possible radiation  $Q$  given by

$$Q_{r \min}^{-1} = (ka)^3 = 6\pi^2 \text{Vol}/\lambda^3, \quad (3)$$

where  $k$  is the wavenumber ( $2\pi/\lambda$ ),  $a$  is the sphere radius, and  $\text{Vol}$  is the volume of the sphere.

Note that equation (3) is given for the minimum  $Q$ . This quantity is less than that derived by Wheeler (1947) in equation (2). Since Wheeler's derivation was based on circuit theory considerations, and included energy stored within the cylindrical volume, this result is reasonable.

The minimum  $Q$  for equal TM and TE modes phased to give circular polarization is half of that given in equation (3). Grimes and Grimes (2002) have derived results for antennas radiating a combination of TE and TM modes. They showed that the minimum  $Q$  depends on

the relative phase of the driven elements. They gave an example of a turnstile antenna that with appropriate phasing has a minimum Q equal to one-third that given in equation (3).

Most antennas are better fit by cylindrical enclosure rather than a sphere. This shape gives two variables, height and diameter. Equations (1) and (3) do not give any insight into the relative effect of height and diameter for this case. Laport (1952) discussed this issue with the implication that VLF antennas should be built as high as possible and then as wide as necessary. The analysis below uses prolate spheroidal coordinates, which somewhat quantifies this relationship.

The approach below is similar to that used by the previous authors. The fields are expanded in prolate spheroidal coordinates and the energy stored and the power radiated are calculated for the region outside an ellipsoid of revolution of a given size. We show that Maxwell's equations are separable in this coordinate system. We believe this is a new result. Unfortunately, the modes are not orthogonal in terms of power. The results are not as general as the previous authors. The results are valid for the minimum radiation Q when only one mode is present. We derive an analytic result for the leading term in the expression for the minimum radiation Q of the lowest order mode.

## GEOMETRICAL RESULTS FOR PROLATE SPHEROIDAL COORDINATES

The relation between Cartesian and prolate spheroidal coordinates is given by

$$x = 0.5 d \cos(\varphi) \sqrt{(u^2 - 1)(1 - \eta^2)} \quad (4)$$

$$y = 0.5 d \sin(\varphi) \sqrt{(u^2 - 1)(1 - \eta^2)} \quad (5)$$

$$z = 0.5 d u \eta \quad (6)$$

This document includes several departures in nomenclature from Flammer (1957). The independent radial coordinate,  $\xi$  in Flammer's book, is represented by  $u$  in this document. The eigenvalue,  $\lambda$  in Flammer's book, for the angular functions, is represented by  $\kappa$ . The variable name,  $\lambda$ , is reserved for the wavelength. The parameter,  $\pi d / \lambda$ , is  $\beta$  rather than  $c$ . The variable,  $c$ , is reserved for the speed of light in vacuum. In one equation in the previous section, the variable,  $\eta$ , was used for the efficiency of the antenna. In all subsequent sections, this parameter is the independent variable equal to the cosine of the polar angle. The parameter,  $d$ , gives the distance between the foci of the ellipse. The equation for an ellipse with semi-minor axis,  $a$ , and semi-major axis,  $b$ , is given by

$$(x/a)^2 + (z/b)^2 = 1. \quad (7)$$

Substituting the expressions for  $x$  and  $z$  from equations (4) and (6) (with  $\varphi$ , the azimuth coordinate set to 0) into equation (7) shows a connection between the minimum value of  $u$  ( $u_m > 1$ ) and the ratio of  $b$  to  $a$ , the aspect ratio. The surface of the ellipsoid of revolution is defined by  $u = u_m$ . The coordinate,  $u$ , varies from  $u_m$  to  $\infty$ . The relation between  $u_m$  and the ratio of  $b$  to  $a$  is given by

$$u_m = (b/a) / \sqrt{(b^2/a^2 - 1)}. \quad (8)$$



The distance between foci of the ellipse,  $d$ , is given by

$$d = 2 b / u_m = 2 a \sqrt{(b^2/a^2 - 1)}. \quad (9)$$

The expressions can be simplified slightly by defining the eccentricity of the ellipse.

An antenna with length,  $L$ , and radius,  $r$ , can be enclosed either by a sphere or by an ellipsoid of revolution. The volume of the antenna is given by  $\pi L r^2$ . The minimum volume of the sphere that surrounds the cylinder is given by

$$\text{Vol (sphere)} = (4\pi/3)((L/2)^2 + r^2)^{3/2} \approx \pi L^3/6 + \pi r^2 L. \quad (10)$$

in which the approximation of a long, thin cylinder ( $L/2 \gg r$ ) has been applied. The "wasted" volume, i.e., the volume of the sphere outside the cylinder is given by  $\pi L^3/6$ . The volume of ellipsoid of revolution that encloses the cylinder is given by

$$\text{Vol (ellipsoid)} = (4\pi/3) r^2 b / (1 - (0.5 \cdot L/b)^2). \quad (11)$$

The value of  $b$  that minimizes the volume is given by  $\sqrt{3} L/2$ . The minimum volume is given by

$$\text{Vol (minimum for ellipsoid)} = \pi r^2 L \sqrt{3}. \quad (12)$$

If  $r$  is much less than  $L/2$ , this minimum volume is much less than that for a sphere. The ellipsoid of revolution fits the long, thin cylinder much more closely than a sphere. In the electromagnetic case, the energy within the ellipsoid should be much less than that in the sphere. This reduced energy implies that the  $Q$  derived in prolate spheroidal coordinates should be less than that for the sphere. Figure 1 shows an example of the geometry. The figure of the ellipse is chosen with parameters to minimize the volume of the ellipsoid of revolution that encompasses the cylinder representing the dipole. The values for the parameters of the cylinder were chosen to accentuate the idea that the sphere and ellipsoid of revolution encompass the dipole.

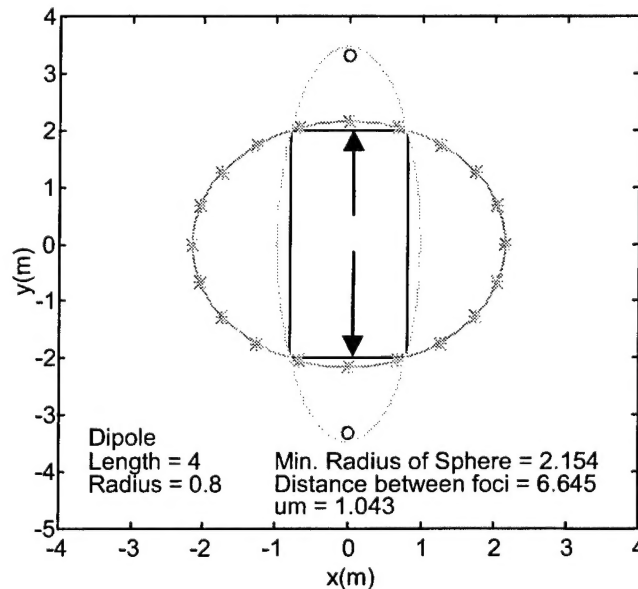


Figure 1. Geometry of dipole exterior.

The parameter,  $d$ , is the distance between foci of the ellipse. Spherical coordinates are recovered by taking the simultaneous limits,  $d \rightarrow 0$  and  $u \rightarrow \infty$ , while the product,  $u d / 2$ , approaches  $r$ . Similarly, the radius,  $a$ , is approached as the product,  $u_m d / 2$ , in which  $u_m$  becomes large and  $d$  becomes small.

## FIELDS EXPRESSED IN PROLATE SPHEROIDAL COORDINATES—VERTICAL POLARIZATION

Maxwell's equations in prolate spheroidal coordinates are most conveniently solved by using the formalism of the Hertz potential. In this method, the fields are the second derivatives of a vector function. The fields are the first derivative of the usual vector and scalar potentials. The addition of a second derivative provides a freedom that allows a derivation of Maxwell's equations as the third derivative of one scalar function. The formalism of Cartan frames and Debye potentials can also be used to derive the results in a coordinate independent way (Cohen and Kegeles, 1974). This formalism, though extremely powerful, provides no additional insight for analyzing Maxwell's equations in the Euclidian spaces appropriate for ordinary conditions.

We express the measured fields,  $\mathbf{E}$  and  $\mathbf{B}$ , in the usual way, in terms of the vector and scalar potentials,  $\mathbf{A}$  and  $\Phi$ , by

$$\mathbf{E} = -\mathbf{A}_t - \text{grad}(\Phi) \quad (13)$$

$$\mathbf{B} = \text{curl}(\mathbf{A}) \quad (14)$$

A bold letter denotes a vector and a subscript denotes differentiation with respect to the variable. The vector and scalar potentials are related to the Hertz potential,  $\mathbf{\Pi}$  by the relation:

$$\mathbf{A} = \mathbf{\Pi}_t / c^2 \quad (15)$$

$$\Phi = -\text{div}(\mathbf{\Pi}) \quad (16)$$

By expressing the vector and scalar potentials in this way, the Lorentz gauge is satisfied identically, and a wave equation results. The symbol,  $c$ , denotes the speed of light in vacuum.

In any curvilinear coordinate system, the divergence, gradient, and curl are related directly to the metric,  $ds^2$ , unit vectors  $\mathbf{u}$ ,  $\mathbf{\eta}$ ,  $\mathbf{\phi}$ , and the metric tensor,  $h_i$ , which has only diagonal coefficients because of the orthogonality of the coordinate system, by the relation,

$$ds^2 = dx^2 + dy^2 + dz^2 = h_1^2 du^2 + h_2^2 d\eta^2 + h_3^2 d\phi^2 \quad (17)$$

$$h_1 = 0.5 d \sqrt{(u^2 - \eta^2)} / \sqrt{(u^2 - 1)} \quad (18a)$$

$$h_2 = 0.5 d \sqrt{(u^2 - \eta^2)} / \sqrt{(1 - \eta^2)} \quad (18b)$$

$$h_3 = 0.5 d \sqrt{(u^2 - 1)} / \sqrt{(1 - \eta^2)} \quad (18c)$$

The independent coordinate,  $u$ , is defined for all  $u$  greater than  $u_m > 1$ . The coordinates,  $\eta$  and  $\phi$ , are defined for all values between  $-1$  to  $1$  and  $0$  to  $2\pi$ , respectively. We will call  $u$  the radial coordinate and  $\eta$  the angular one.

The gradient is given by

$$\text{grad}(\Phi) = \underline{u} \Phi_u / h_1 + \underline{\eta} \Phi_\eta / h_2 + \underline{\varphi} \Phi_\varphi / h_3 \quad (19)$$

The divergence is given by

$$\text{div}(\mathbf{A}) = ((h_2 h_3 \underline{u} \cdot \mathbf{A})_u + (h_1 h_3 \underline{\eta} \cdot \mathbf{A})_\eta + (h_1 h_2 \underline{\varphi} \cdot \mathbf{A})_\varphi) / (h_1 h_2 h_3) \quad (20)$$

The curl is given by

$$\begin{aligned} \text{curl}(\mathbf{A}) = & \underline{u} ((h_3 \underline{\varphi} \cdot \mathbf{A})_\eta - (h_2 \underline{\eta} \cdot \mathbf{A})_\varphi) / (h_2 h_3) + \underline{\eta} ((h_1 \underline{u} \cdot \mathbf{A})_\varphi - (h_3 \underline{\varphi} \cdot \mathbf{A})_u) / (h_3 h_1) + \\ & \underline{\varphi} ((h_2 \underline{\eta} \cdot \mathbf{A})_u - (h_1 \underline{u} \cdot \mathbf{A})_\eta) / (h_1 h_2) \end{aligned} \quad (21)$$

Specialization to a vertically polarized system occurs if (Flammer, 1957, p. 70):

$$\mathbf{A} = (\underline{u} \eta / h_1 + \underline{\eta} u / h_2) 0.5 d e_t / c^2 \quad (22a)$$

$$\Phi = -0.5 d \text{div}((\underline{u} \eta / h_1 + \underline{\eta} u / h_2) \mathbf{e}) \quad (22b)$$

The choice of the functional form for  $\mathbf{A}$  and  $\Phi$  in terms of  $u$  and  $\eta$  comes from the definition of a unit vector in the  $z$  direction, ( $\underline{z}$ ). The vector potential should be proportional to  $\underline{z}$  so that the electric field also has this polarization. We also specialize to omnidirectional antennas by requiring that no quantity depends upon the azimuth coordinate,  $\varphi$ . The dependent variable,  $e$ , is a scalar quantity and an arbitrary function of  $t$ ,  $u$ , and  $\eta$ . The current density,  $\mathbf{J}$ , in Maxwell's equations must also be in the vertical direction so that we also have

$$\mathbf{J} = (\underline{u} \eta / h_1 + \underline{\eta} u / h_2) 0.5 d v_t / c^2 \quad (22c)$$

This equation introduces the function  $v$  and its time derivative. Inputting the assumptions presented in equations (22a), (22b), and (22c) into the definitions of the gradient, divergence, and curl leads to two identical equations for the time and space variation of  $e$ . After integration with respect to  $t$ , the one independent equation is

$$(\mu_0 v - e_{tt} / c^2) d^2 (u^2 - \eta^2) / 4 = -(u^2 - 1) e_{uu} - (1 - \eta^2) e_{\eta\eta} + 2\eta e_\eta - 2u e_u \quad (23)$$

Both components of the equation for  $e$  result in the same equation. If a time dependence for  $e$  is given by  $\exp(-j \omega t)$  and the current density is zero, the solution to equation (23) is given by the product of eigenfunctions of spheroidal wave equations for  $m = 0$ :

$$0 = ((1 - \eta^2) S^{(n,0)}_\eta)_\eta + (\kappa^{(n,0)} - \beta^2 \eta^2) S^{(n,0)} \quad (24a)$$

$$0 = ((u^2 - 1) R^{(n,0)}_u)_u + (\beta^2 u^2 - \kappa^{(n,0)}) R^{(n,0)} \quad (24b)$$

The separation constant,  $\kappa^{(n,0)}$ , is the eigenvalue of equation (24a) adjusted so that the solution is regular for all  $\eta$  between -1 and 1. The eigenvalue is a function of the frequency,  $\omega$ , and its related parameter,  $\beta = \omega d / (2c)$ . The Appendix describes the numerical solution of equations (24a) and (24b). Tables for the eigenvalues and normalized eigenfunctions are described in Flammer (1957) and Abramowitz and Stegun (1965). In all subsequent work, we assume simple exponential time dependence. Figure 2 compares the numerical results for solving the eigenvalue equation with those in Flammer (1957). The comparison always agrees with every digit listed in Flammer's book. The Appendix presents the methods used in computing the eigenvalue, eigenfunctions, or the angular and radial equations.

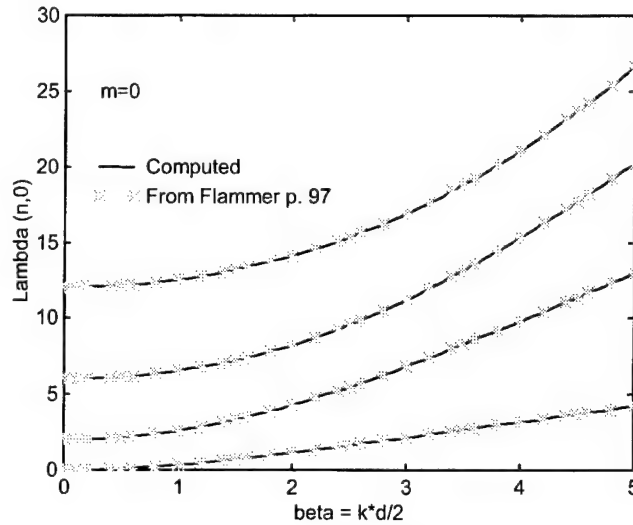


Figure 2. Eigenvalue computation.

In terms of  $e$ , the electric field,  $E$ , and the magnetic field,  $B$ , are given by:

$$\mathbf{E} = \eta \underline{\mathbf{u}} \{ \beta^2 e + ((e(u^2-1))_u / (u^2-\eta^2))_u + (u (e(1-\eta^2))_\eta / (u^2-\eta^2))_u / \eta \} (4/d^2) \sqrt{(u^2-1)} / \sqrt{(u^2-\eta^2)} + \underline{\mathbf{u}} \{ \beta^2 e + (\eta (e(u^2-1))_u / (u^2-\eta^2))_\eta / u + ((e(1-\eta^2))_\eta / (u^2-\eta^2))_\eta \} (4/d^2) \sqrt{(1-\eta^2)} / \sqrt{(u^2-\eta^2)} \quad (25a)$$

$$\mathbf{B} = -4 j \beta \underline{\mathbf{u}} (ue_u - \eta e_\eta) \cdot \{ cd^2(u^2-\eta^2) \}^{-1} \sqrt{(u^2-1)} \sqrt{(1-\eta^2)} \quad (25b)$$

These expressions are used in the subsequent section to calculate the Poynting vector. We will eventually expand the dependent variable,  $e$ , in a series of the complete set of eigenfunctions. The radial function,  $R$ , will be specialized to deal only with outgoing waves.

## FIELDS EXPRESSED IN PROLATE SPHEROIDAL COORDINATES—HORIZONTAL POLARIZATION

Calculation of the fields and the Poynting vector are much simpler for the case of horizontal polarization than for vertical polarization. This calculation is important if circular polarization is required. The Hertz potential has only one component,

$$\mathbf{A} = \underline{\mathbf{u}} e_t / c^2 \quad (26a)$$

$$\Phi = 0 \quad (26b)$$

$$\mathbf{J} = \underline{\mathbf{u}} v_t \quad (26c)$$

Applying the formulas leads to one non-trivial equation:

$$-\mu_0 v d^2 (u^2 - \eta^2)/4 = \beta^2 e (u^2 - \eta^2) + (u^2 - 1)e_{uu} + 2ue_u - e/(u^2 - 1) + (1 - \eta^2)e_{\eta\eta} - 2\eta e_\eta - e/(1 - \eta^2) \quad (27)$$

The solution to equation (27) leads to the set of eigenfunctions for the spheroidal wave equations for  $m = 1$ :

$$0 = ((1-\eta^2)S^{(n,1)}_{\eta})_{\eta} + (\kappa^{(n,1)} - \beta^2 \eta^2 - 1/(1-\eta^2)) S^{(n,1)} \quad (28a)$$

$$0 = ((u^2-1)R^{(n,1)}_u)_u + (\beta^2 u^2 - \kappa^{(n,1)} - 1/(u^2-1))R^{(n,1)} \quad (28b)$$

As in the vertical polarization case, only outgoing waves are chosen. In the language of prolate spheroidal coordinates, the radial function will be asymptotic to

$$R^{(n,1)} \approx (1/(\beta u)) \exp(j(\beta u - (n+1)\pi/2)) \text{ as } u \rightarrow \infty \quad (29)$$

This expression is called the radial function of the third kind. The complex conjugate of  $R$  is called the radial function of the fourth kind.

The expressions for the electric and magnetic fields are given by

$$\mathbf{E} = (2\beta/d)^2 \underline{\mathbf{e}} \quad (30a)$$

$$\mathbf{B} = -4j(\beta/(cd^2)) \{ \underline{\mathbf{u}} ((1-\eta^2)\mathbf{e}_{\eta} - \eta\mathbf{e}/(1-\eta^2))\sqrt{(1-\eta^2)}\sqrt{(u^2-\eta^2)} - \underline{\mathbf{u}} ((u^2-1)\mathbf{e}_u + \mathbf{e}/(u^2-1))\sqrt{(u^2-1)}/\sqrt{(u^2-\eta^2)} \} \quad (30b)$$

## EXPRESSIONS FOR POYNTING VECTOR— VERTICAL AND HORIZONTAL POLARIZATIONS

Two surfaces are important for this problem. The first is the one that surrounds the antenna, which is defined by  $u = u_m > 1$ . The second is located at distances that are large when compared to a wavelength ( $u \gg 1$ ). Only that component of the Poynting vector in the  $u$  direction will be important for describing radiation. The other component is tangential to these surfaces.

The expression for the Poynting vector,  $\mathbf{S}$ , the power per unit area through a surface, is given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{B}^*/(2\mu_0) \quad (31)$$

The power through the surface defined by  $u = \text{constant}$  is given by

$$P(u) = 2\pi \int_{-1}^1 h_2 h_3 \underline{\mathbf{u}} \cdot (\mathbf{E} \times \mathbf{B}^*)/(2\mu_0) d\eta \quad (32)$$

Inserting the expressions for  $h_2$ ,  $h_3$ ,  $\mathbf{E}$ , and  $\mathbf{B}$  results in

$$P(u) = (2\pi j \omega u (u^2-1)/\mu_0 dc^2) \int d\eta (1-\eta^2)/(u^2-\eta^2)^2 \{ \beta^2 u^2 \mathbf{e} + (u^2-1)(u^2+\eta^2)\mathbf{e}_u/(u(u^2-\eta^2)) + \eta \mathbf{e}_{\eta} (u^2-1)/u + \mathbf{e}_{\eta\eta} (1-\eta^2) - 2\eta \mathbf{e}_{\eta} \beta^2 \eta^2 \mathbf{e} + 2\eta \mathbf{e}_{\eta} (1-\eta^2)/(u^2-\eta^2) \} (\mathbf{u} \mathbf{e}_u^* - \eta \mathbf{e}_{\eta}^*) \quad (33)$$

In equation (33), the assumption of a time dependence of  $\exp(-i\omega t)$  has been made. If we expand the dependent variable,  $\mathbf{e}$ , in a series of the complete set,  $S^{(n,0)}$  as

$$\mathbf{e} = \sum_{n=0}^{\infty} \mathbf{e}_n S^{(n,0)}(\beta, \eta) R^{(n,0)}(\beta, u) \quad (34)$$

equation (33) then becomes

$$P(u) = (2\pi j \omega u (u^2 - 1) / \mu_0 d c^2) \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} e_n e_l^* \int d\eta (1 - \eta^2) / (u^2 - \eta^2)^2 \{ \beta^2 u^2 S^{(n,0)} R^{(n,0)} - \kappa_n S^{(n,0)} R^{(n,0)} + (u^2 - 1)(u^2 + \eta^2) / (u(u^2 - \eta^2) S^{(n,0)} R^{(n,0)}_u + \eta(u^2 - 1) S^{(n,0)}_\eta R^{(n,0)}_u / u + 2\eta(1 - \eta^2) R^{(n,0)} S^{(n,0)}_\eta / (u^2 - \eta^2) \} \{ u S^{(l,0)} R^{(l,0)*}_u - \eta R^{(l,0)*} S^{(l,0)}_\eta \} \quad (35)$$

In the limit as  $u \rightarrow \infty$ , the radial function take their asymptotic forms (equation (29)) and the total power for vertically polarized radiation becomes

$$P(u \rightarrow \infty) = (4\pi \beta^2 / d^2 c \mu_0) \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} e_n e_l^* \int d\eta (1 - \eta^2) S^{(n,0)} S^{(l,0)} \quad (36)$$

Only if only one mode is excited is the total power positive definite. Unfortunately, the orthogonality of the  $S$  in the weight function of equation (36) cannot be assumed. The presence of terms with  $u$  in the denominator makes the orthogonality doubtful. The situation for horizontally polarized radiation is much simpler. Inserting the expressions for the electromagnetic field into the expressions for the power leads to

$$P(u) = (2\pi j \omega^3 d / 4c^4 \mu_0) \int d\eta e ((u^2 - 1) e_u^* + u e^*) \quad (37)$$

Expanding the dependent variable,  $e$ , in a series of  $S^{(n,1)}$  leads to

$$P(u) = (4\pi j \beta^3 / d^2 c \mu_0) \sum_{n=1}^{\infty} |e_n|^2 R^{(n,1)} \{ u R^{(n,1)*} + (u^2 - 1) R^{(n,1)*}_u \} \int d\eta S^{(n,1)2} \quad (38)$$

The double sum reduces to a single one because of the orthogonality of the  $S$  over the interval. In the limit of large  $u$ , this expression for horizontal polarized radiation reduces to

$$\text{Power}(u \rightarrow \infty) = (4\pi \beta^2 / d^2 c \mu_0) \sum_{n=1}^{\infty} |e_n|^2 \int d\eta S^{(n,1)2} \quad (39)$$

The power at infinity is a positive definite function. The expression in equation (39) goes directly over to the analysis given in Chu (1948) and Grimes and Grimes (1995) for determining the expression for the  $Q$  of the antenna.

## APPROXIMATION OF SMALL ELECTRICAL SIZE

If small electrical size is approximated, some formulas can be simplified. The parameter that relates to electrical size is  $\beta = \omega d / (2c)$ . Small  $\beta$  is equivalent to small electrical size. Most of the formulas are functions of the square of  $\beta$ .

The equation for the angular eigenfunction in prolate spheroidal coordinates is

$$0 = ((1 - \eta^2) S^{(n,m)}_\eta)_\eta + (\kappa^{(n,m)} - \beta^2 \eta^2 - m^2 / (1 - \eta^2)) S^{(n,m)} \quad (40)$$

This equation is almost identical to that for associated Legendre polynomials:

$$0 = ((1 - \eta^2) P_n^m)_\eta + (n(n+1) - m^2 / (1 - \eta^2)) P_n^m \quad (41)$$

Only the term  $\beta^2 \eta^2$  is different. This term should be considered a small correction for an electrically small antenna. Because of the standard notation for Legendre polynomials, we

have put the subscript, 1, on  $P_l^m$ . This should not cause confusion with the derivative with respect to a variable.

Abramowitz and Stegun (1965, p. 334) give the recursion relation for the Legendre polynomials:

$$(n-m+1) P_{n+1}^m = (2n+1) \eta P_n^m - (n+m) P_{n-1}^m \quad (42)$$

Using this recursion relation, the orthogonality of the Legendre polynomials, and the integral of the square (determined to the normalization),

$$\int_{-1}^1 d\eta P_n^m P_n^m = 2 (n+m)! / ((2n+1) (n-m)!), \quad (43)$$

we determined the following integrals:

$$\int_{-1}^1 d\eta \eta P_n^m P_{n+1}^m = 2 (n+1+m) (n+m)! / ((2n+1) (2n+3) (n-m)!) \quad (44a)$$

$$\int_{-1}^1 d\eta \eta P_n^m P_{n-1}^m = 2 (n-m) (n+m)! / ((2n-1) (2n+1) (n-m)!) \quad (44b)$$

$$\int_{-1}^1 d\eta \eta^2 P_n^m P_n^m = 2(2n^2+2n-1-2m^2)(n+m)! / ((2n+1)(2n-1)(2n+3)(n-m)!) \quad (44c)$$

$$\int_{-1}^1 d\eta \eta^2 P_n^m P_{n+2}^m = 2(n+2+m)(n+1+m)(n+m)! / ((2n+1)(2n+3)(2n+5)(n-m)!) \quad (44d)$$

$$\int_{-1}^1 d\eta \eta^2 P_n^m P_{n-2}^m = 2(n-m)(n-1-m)(n+m)! / ((2n-1)(2n-3)(2n+1)(n-m)!) \quad (44e)$$

We seek to determine the eigenfunctions and eigenvalues of equation (40) through first order in  $\beta^2$ . Most relations are well determined by zeroth order. We seek to understand the effect of finite but small size upon these relations. Accordingly, we express the eigenvalue,  $\kappa^{(n,m)}$ , as a two-term expansion,

$$\kappa^{(n,m)} = n(n+1) + y^{(n,m)} \beta^2, \quad (45)$$

and the eigenfunction,  $S^{(n,m)}$ , as a three-term expansion,

$$S^{(n,m)} = d_0^{(n,m)} P_n^m + \beta^2 (d_1^{(n,m)} P_{n+2}^m + d_2^{(n,m)} P_{n-2}^m) \quad (46)$$

Only two of the terms contribute in equation (46), if  $n$  is 0 or 1. Substituting equations (45) and (46) into equation (40) and using the various results above leads to

$$\kappa^{(n,m)} = n(n+1) + (2n^2+2n-1-2m^2) \beta^2 / ((2n-1)(2n+3)) \quad (47)$$

and

$$d_1^{(n,m)} = -(n+2-m)(n+1-m) d_0^{(n,m)} / (2(2n+3)^2(2n+1)) \quad (48a)$$

$$d_2^{(n,m)} = (n+m)(n-1+m) d_0^{(n,m)} / (2(2n-1)^2(2n+1)) \quad (48b)$$

The overall factor,  $d_0^{(n,m)}$ , is determined by the scale factor of the Legendre polynomials (Flammer, 1957, p. 21):

$$S^{(n,m)}(\beta, \eta = 0) = P_n^m(\eta = 0) = (-1)^{((n-m)/2)} (n+m)! / (2^n ((n-m)/2)! ((n+m)/2)!) \quad (49a)$$

if  $n-m$  is even, or

$$S^{(n,m)}(\beta, \eta = 0)_{\eta} = P_n^m(\eta = 0)_{\eta} = (-1)^{((n-m-1)/2)} (n+m+1)! / (2^n ((n-m-1)/2)! ((n+m+1)/2)!), \quad (49b)$$

where the derivative is evaluated at  $\eta = 0$  for  $n-m$  odd. The final result for  $d_0^{(n,m)}$  is complicated and will not be written here. Because the eigenfunctions are expressed as three Legendre polynomials, they are easily evaluated. The eigenvalue is a simple function of the parameters,  $n$ ,  $m$ , and  $\beta$ .

The radial function is somewhat more difficult to analyze. In general, the radial function is given as an integral over  $\eta$  of a weighted product of spherical Hankel functions and the eigenfunctions for the angular equation. The general expression is given by

$$R^{(n,m)}(\beta, u) = j^{n-m} (2m+1) (u^2 - 1)^{m/2} / (2^{m+1} m! d^{mn}_0(\beta)) \int_{-1}^1 d\eta h^{(1)}_m[\beta(u^2 + \eta^2 - 1)^{1/2}] (1 - \eta^2)^{m/2} S^{(n,m)}(\beta, \eta) (u^2 + \eta^2 - 1)^{-m/2} \quad (50a)$$

for  $n-m$  even, or

$$R^{(n,m)}(\beta, u) = j^{n-m-1} (2m+3) u (u^2 - 1)^{m/2} / (2^{m+1} m! d^{mn}_1(\beta)) \int_{-1}^1 d\eta \eta h^{(1)}_{m+1}[\beta(u^2 + \eta^2 - 1)^{1/2}] (1 - \eta^2)^{m/2} S^{(n,m)}(\beta, \eta) (u^2 + \eta^2 - 1)^{-(m+1)/2} \quad (50b)$$

for  $n-m$  odd.

The function  $h^{(1)}_m$  is the spherical Hankel function of the first kind. In the limit of small argument, the Hankel function is dominated by the imaginary part, a spherical Neuman function. For small values of the argument, the Hankel function is given by Abramowitz and Stegun (1965, p. 437):

$$h^{(1)}_m(z) = -j \cdot 1 \cdot 3 \cdot 5 \dots (2m-1) z^{-(m+1)} [1 - (z^2/2)/(1!(1-2m)) + (z^2/2)^2/(2!(1-2m)(3-2m)) \dots] \quad (51)$$

In the limit of small electrical size, the parameters  $d^{mn}_0$  and  $d^{mn}_1$  become approximately equal to 1 and the function  $S^{(m,n)}$  becomes an associated Legendre polynomial. The integral in equation (50a) is easily done for  $m = 0$  and  $n = 0$ . The result is

$$R^{(0,0)}(\beta, u) = -j/(2\beta) \ln((u+1)/(u-1)) \quad (52)$$

The integral can also be evaluated for  $m$  and  $n$  less than 2 in terms of elementary functions. The results can probably be extended to almost any order by using differentiation under the integral sign and the following identity:

$$\int d\eta (s + w\eta^2)^{-1/2} = w^{-1/2} \ln(\eta w^{1/2} + (s + w\eta^2)^{1/2}) \quad (53)$$

Differentiation with respect to  $s$  in equation (53) leads to higher powers of  $(s + w\eta^2)$  in the denominator. Differentiation with respect to  $w$  leads to higher even powers of  $\eta$  in the numerator and higher powers of  $(s + w\eta^2)$  in the numerator.

Comparisons with numerical evaluation of the integral, equation (50a), and solution of the original differential equation (24b) show that this expression and that for the derivative with respect to  $u$  are accurate to three digits if the value of  $u$  is less than  $0.3/\beta$ . Accuracy for larger  $u$  is extended if the form chosen is given by

$$R^{(0,0)}(\beta, u) = -(j/2\beta) \exp(j\beta u) \ln((u+1)/(u-1)) \quad (54)$$



Figures 3, 4, 5, 6, and 7 compare the numerical evaluation of the differential equation and the analytic function in equation (53).

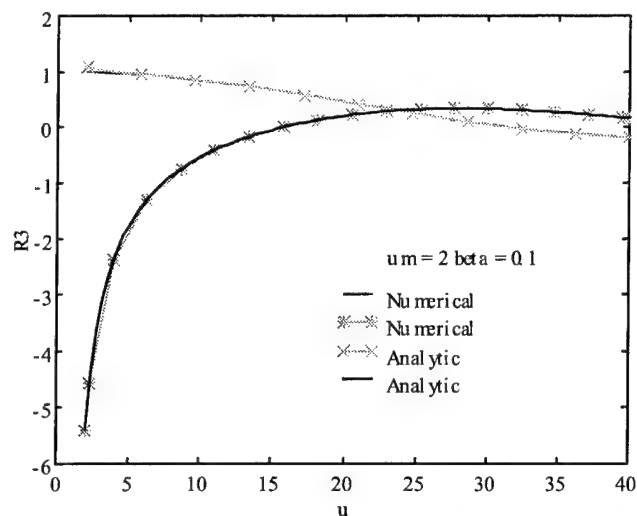


Figure 3. Numerical and analytic approximation comparison for  $R(3)$ .

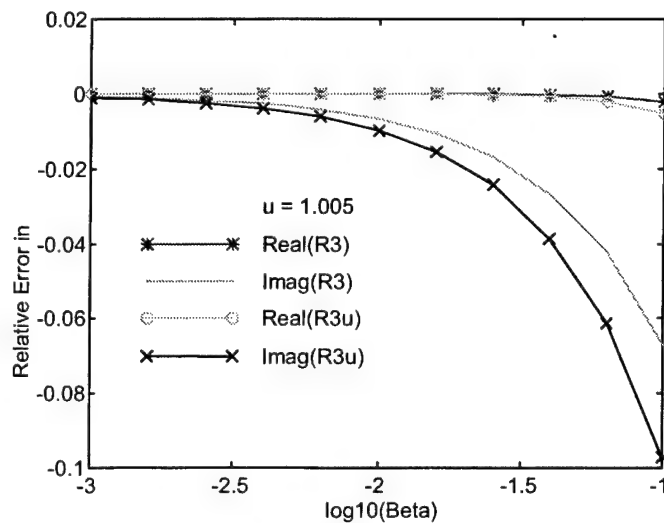


Figure 4. Error in computing real and imaginary parts from analytic formula ( $u = 1.005$ ).

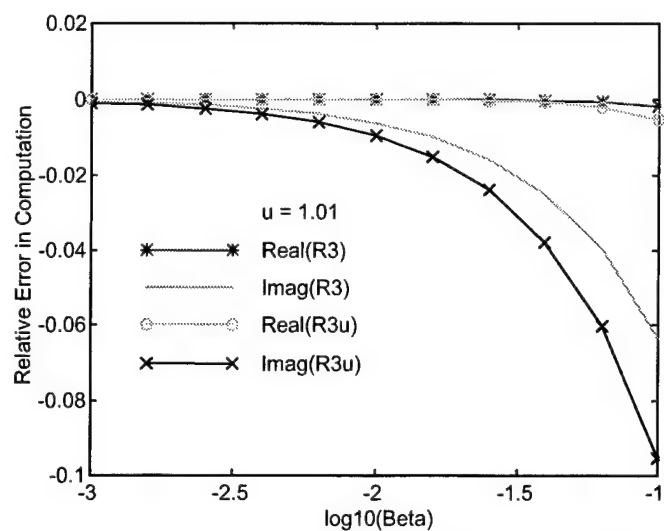


Figure 5. Error in computing real and imaginary parts from analytic formula ( $u = 1.01$ ).

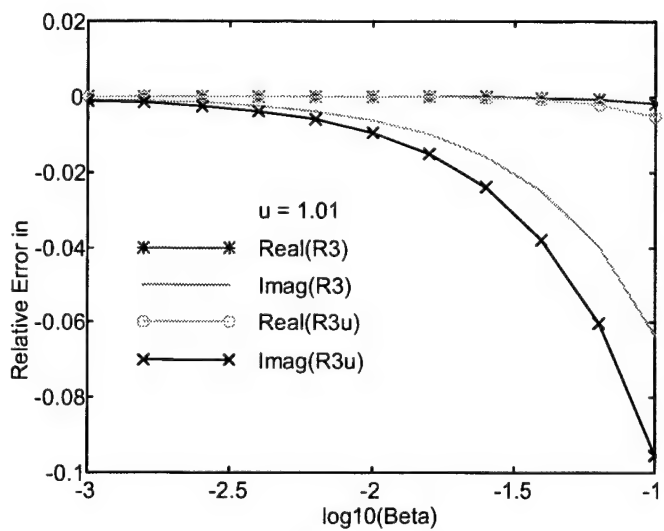


Figure 6. Error in computing real and imaginary parts from analytic formula ( $u = 1.05$ ).

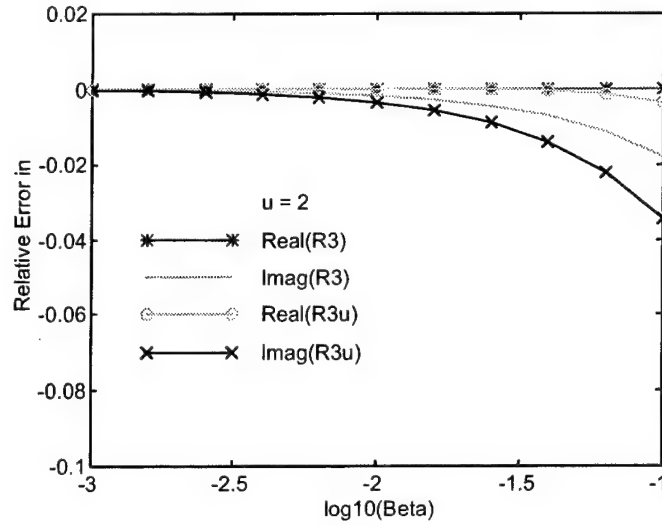


Figure 7. Error in computing real and imaginary parts from analytic formula ( $u = 2$ ).

In these figures, we have evaluated the relative accuracy while keeping  $u$  constant and varying  $\beta$  and varying  $u$  while keeping  $\beta$  constant. This form for the radial function will be used in subsequent sections. The accuracy is sufficient for our purposes.

## ANTENNA Q

The  $Q$  of an electronic device is equal to the ratio of bandwidth of operation to the center frequency. Antenna bandwidth has multiple definitions. They include full width at half maximum of the gain versus frequency, frequency span over which the return loss is 10 dB lower than its maximum, or the span over which the voltage standing wave ratio (VSWR) is lower than a certain number. Each definition has its use. We will use the relation between power,  $P(u)$ , radiated through a surface, and the energy,  $W$ , stored within the volume bounded by the surface.

The power has already been evaluated for an arbitrary distribution of modes. The expression for  $Q$  is given by

$$Q = 2 \omega W_{\text{stored}} / P(u \rightarrow \infty) \quad (55)$$

This equation is a common expression for  $Q$  when dealing with antennas. As Chu (1948) shows, this expression can be related to the parameters of a circuit. In general, the expression for the energy within the field is determined from Maxwell's equation by

$$W_{\text{total}} = \int dV (\epsilon_0 \mathbf{E} \cdot \mathbf{E}^* / 4 + \mathbf{B} \cdot \mathbf{B}^* / (4\mu_0)) \quad (56)$$

The volume element in prolate spheroidal coordinates is given by

$$\int dV = \int h_1 h_2 h_3 \, du d\eta d\phi = 2\pi (d/2)^3 \int (u^2 - \eta^2) \, du d\eta \quad (57)$$

The integral above is evaluated over all space external to the volume. The idea is that no energy is stored close to the antenna. One reason for using prolate spheroidal coordinates is that there is less excluded volume.

The variable,  $u$ , is between  $u_m$  and  $\infty$ , and  $\eta$  is between  $-1$  and  $1$ . The stored energy is the difference between the total energy given in equation (56) and that radiated away. The energy per unit volume radiated away is the Poynting vector divided by the speed of light. At infinity, subtracting the value of the Poynting vector cancels the asymptotic value of the total energy so that the total stored energy is a finite value.

The stored energy can be written as

$$W_{\text{stored}} = 2\pi (d/2)^3 \int_{u_m}^{\infty} \int_{-1}^1 (u^2 - \eta^2) du d\eta (\epsilon_0 \mathbf{E} \cdot \mathbf{E}^* / 4 + \mathbf{B} \cdot \mathbf{B}^* / (4\mu_0) - \mathbf{u} \cdot \mathbf{E} \times \mathbf{B}^* / (2\mu_0 c)) \quad (58)$$

The expression in equation (58) is a variation of the expression in Collin and Rothschild (1964) or McLean (1996) for an antenna enclosed by a sphere of radius,  $a$ . Using the expressions for  $\mathbf{E}$  and  $\mathbf{B}$  already given in previous sections, we find

$$W_{\text{stored}} = \pi \epsilon_0 d^{-1} \int_{u_m}^{\infty} du \int_{-1}^1 d\eta (\eta^2(u^2-1) \|Q_1\|^2 / (u^2 - \eta^2)^4 + u^2(1 - \eta^2) \|Q_2\|^2 / (u^2 - \eta^2)^4 + \beta^2(u^2 - 1)(1 - \eta^2) \|Q_3\|^2 / (u^2 - \eta^2)^2 - 2jQ_2Q_3^* u(1 - \eta^2) \sqrt{u^2 - 1} / (u^2 - \eta^2)^{7/2}), \quad (59)$$

where

$$Q_1 = R S (\kappa - \beta^2 \eta^2) + (1 - \eta^2) S_{\eta} / \eta (u R_u - (u^2 + \eta^2) R / (u^2 - \eta^2)) - 2u(u^2 - 1) S R_u / (u^2 - \eta^2) \quad (60a)$$

$$Q_2 = (\beta^2 u^2 - \kappa) R S + (u^2 - 1) R_u / u (\eta S_{\eta} + (u^2 + \eta^2) S / (u^2 - \eta^2)) + 2\eta R (1 - \eta^2) S_{\eta} / (u^2 - \eta^2) \quad (60b)$$

and

$$Q_3 = S u R_u - R \eta S_{\eta} \quad (60c)$$

In terms of the dependent variables given above, the power radiating through the surface at  $u = u_m$  is given by

$$P = 4\pi j \beta u (u^2 - 1) / (d^2 c \mu_0) \int_{-1}^1 d\eta (1 - \eta^2) Q_2 Q_3^* / (u^2 - \eta^2)^3 \quad (61)$$

In general, the integrals must be done numerically, even when the radial and angular functions are given in terms of simple analytical functions.

Fortunately, the integrals can be done in equation (59) for the leading term in  $\beta$ . This term will lead to an inverse cube relationship between  $Q$  and the dimensionless wave number. This relationship is important in its own right. The result also gives confidence in the numerical results presented.

From equation (54) we use the expression for the first derivative of the radial function:

$$R_u(u) = (.5 \ln((u+1)/(u-1)) + j/(\beta(u^2-1))) \exp(j\beta u) \quad (62)$$

The small  $\beta$  limit of the eigenvalue and the angular eigenfunction are given by:

$$\lambda = \beta^2 / 3 \quad (63a)$$

$$S = 1 - \beta^2 \eta^2 / 6 \quad (63b)$$

The expressions above are convenient. Any derivative of  $S$  with respect to the angular coordinate leads to an extra power of  $\beta^2$ . The eigenvalue is also proportional to this extra factor. The eigenfunction can usually be set equal to 1 with little error. The leading term occurs from the factor of  $\beta$  in the denominator of the expression for  $R_u$ . The similar term in the denominator of  $R$  in equation (54) is always multiplied by  $\beta^2$ .

The expression for the stored energy from equation (59) is given by

$$W_{\text{stored}} = (\pi\epsilon_0/d\beta^2) \int_{u_m}^{\infty} du \int_{-1}^1 d\eta [4\eta^2 u^2 (u^2 - 1) + (1 - \eta^2)(u^2 + \eta^2)^2] (u^2 - \eta^2)^{-4} \quad (64)$$

The integrals are done in a straightforward manner. The book of integrals given in Dwight (1961, pp. 35–36) is one example. The final result is

$$W_{\text{stored}} = (\pi\epsilon_0/d\beta^2) [u_m/(u_m^2 - 1) - 0.5/u_m + 0.25(u_m^{-2} - 1)\ln((u_m + 1)/(u_m - 1))] \quad (65)$$

In the same limit the total power is given by

$$P_{\text{total}} = 16\pi\beta^2/(3d^2c\mu_0) \quad (66)$$

Equation (65) represents one of the more important results of this document. Because  $Q$  is the product of the radial frequency,  $\omega = 2c\beta/d$  and  $W_{\text{stored}}/P_{\text{total}}$ , the final result for  $Q$  is

$$Q = 3/(4\beta^3) [u_m/(u_m^2 - 1) - 0.5/u_m + 0.25(u_m^{-2} - 1)\ln((u_m + 1)/(u_m - 1))] \quad (67)$$

Figure 8 shows the result for  $Q$  given for the lowest mode as a function of  $\beta$  for six values of the minimum  $u_m$ . For values of  $\beta$  less than 0.01, the expression for  $Q$  given by equation (67) is confirmed by doing the integral in equation (64) numerically to four digits.

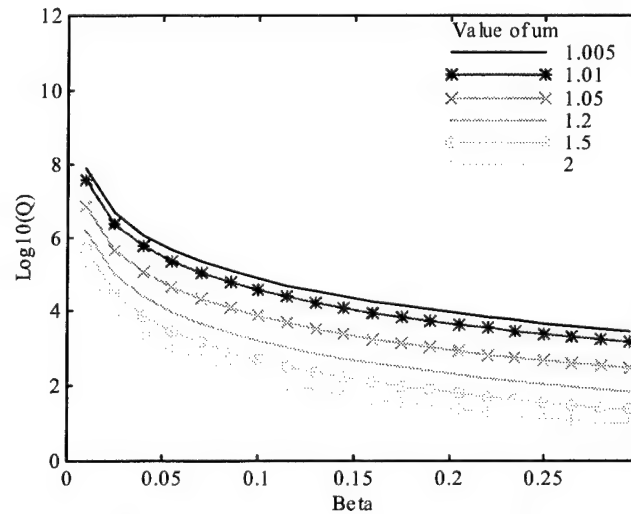


Figure 8. Logarithm of  $Q$  versus Beta.

Table 1 lists the logarithm to base 10 of the value of  $Q$  versus the dimensionless wave number for six values of the shape factor,  $u_m$ . If this shape factor gets close to 1, the shape becomes more like a line. As this factor increases, the shape becomes more spherical.

Table 1. Logarithm to the base 10 of Q versus  $\beta$  for six values of  $u_m$ .

$\beta \backslash u_m$	1.005	1.010	1.050	1.200	1.500	2.000
0.010	7.8761	7.5718	6.8610	6.2029	5.6834	5.1989
0.025	6.6822	6.3779	5.6671	5.0093	4.4902	4.0065
0.040	6.0697	5.7654	5.0547	4.3972	3.8788	3.3966
0.055	5.6546	5.3504	4.6398	3.9827	3.4654	2.9854
0.070	5.3401	5.0359	4.3255	3.6691	3.1532	2.6759
0.085	5.0869	4.7827	4.0725	3.4169	2.9027	2.4288
0.100	4.8747	4.5706	3.8606	3.2060	2.6938	2.2240
0.115	4.6922	4.3880	3.6784	3.0249	2.5150	2.0497
0.130	4.5320	4.2278	3.5186	2.8663	2.3591	1.8988
0.145	4.3891	4.0850	3.3761	2.7253	2.2210	1.7661
0.160	4.2602	3.9562	3.2477	2.5985	2.0973	1.6483
0.175	4.1428	3.8388	3.1308	2.4832	1.9855	1.5427
0.190	4.0349	3.7310	3.0235	2.3778	1.8837	1.4474
0.205	3.9350	3.6312	2.9243	2.2806	1.7905	1.3609
0.220	3.8422	3.5384	2.8321	2.1905	1.7045	1.2819
0.235	3.7553	3.4516	2.7459	2.1066	1.6250	1.2095
0.250	3.6737	3.3700	2.6651	2.0282	1.5511	1.1428
0.265	3.5967	3.2931	2.5889	1.9546	1.4822	1.0812
0.280	3.5238	3.2204	2.5169	1.8853	1.4178	1.0241
0.295	3.4546	3.1513	2.4487	1.8198	1.3573	0.9710

## DISCUSSION

The value of  $u_m$  is related to the aspect ratio of the ellipsoid of revolution. This relationship is displayed in equation (8). This number is independent of the volume and relates only to the geometry of the basic ellipse. To relate the  $Q$  to the total volume,  $V$ , and to the wave number,  $k$ , of the radiation, is straightforward. The total volume of an ellipsoid of revolution with major axis length,  $2b$ , and minor axis length,  $2a$ , is given by

$$Vol = 4\pi a^2 b / 3 = 4\pi d^3 (u_m^2 - 1) u_m / 24 \quad (68a)$$

Because a factor of  $\beta^3 = (k d/2)^3$  is in the denominator of the expression for  $Q$ , the volume can be extracted from the final result. This extraction will allow the shape-dependent effects to be separated from the volume. The result is as follows:

$$Q = (\pi/2k^3 Vol) u_m (u_m^2 - 1) [u_m / (u_m^2 - 1) - 0.5/u_m + 0.25(u_m^2 - 1) \ln((u_m + 1)/(u_m - 1))] \quad (68b)$$

As shown in Figure 8 and equation (67), for a given value of  $\beta$ , the  $Q$  is a monotonically decreasing function of the shape factor,  $u_m$ . This decrease is largely due to the change in the volume. As  $u_m$  approaches 1, the value of  $Q$  increases. This increase is reasonable. A long, thin dipole has a higher  $Q$  than a broad one. The broad dipole has many paths in which to complete the circuit. The resonance width is broadened with increasing lateral extent. The value of  $Q$  is a decreasing function of  $\beta$ .

If the shape and the wavenumber remain constant and the value of the volume is decreased, the value of  $Q$  increases. For constant volume as the value of  $u_m$  increases, the value of  $Q$  also increases. The volume is becoming more spherical. Storage of energy in the volume increases. For constant volume as  $u_m$  approaches 1 (a very thin dipole), the value of  $Q$  approaches zero.

The inverse cube dependence of  $Q$  on the wavenumber is typical for an electrically small antenna. Many authors, starting with Chu (1948), have calculated this dependence as the first term in a power series. Grimes and Grimes (1995) have provided a more accurate calculation that reduces to such a power series in the appropriate limit. The coefficient multiplying the  $k^{-3}$  is somewhat smaller. For a small dipole surrounded by a sphere of radius,  $a$ , the value of  $Q$  (McLean, 1996, among many others) is

$$Q_{\text{sphere}} = (k a)^{-3} + (k a)^{-1} \quad (69)$$

For  $u_m = 2$ , the first term for an ellipsoid of revolution is given by

$$Q_{\text{ellipsoid}} = 0.158 \cdot (2/k d)^3 \quad (70)$$

The coefficient of the  $k^{-3}$  is much smaller for the ellipsoid with a particular aspect ratio than for the dipole surrounded by a sphere. The sphere surrounds a much larger volume. This "wasted" volume contributes significantly to the  $Q$  in this theoretical limit.

Figure 9 compares Chu's results for the sphere (which has an aspect ratio of 1) with those of the prolate spheroid. The leading terms of the sphere and the ellipsoid of revolution are proportional to the cube of the wavelength and the inverse of the volume. Multiplying the value of  $Q$  by the product of the volume divided by the cube of the wavelength allows a presentation that depends only upon shape. The value for the ellipsoid of revolution becomes that for the sphere as the aspect ratio gets close to 1. If the volume is held constant, the value

of  $Q$  for the ellipsoid is always less than that for the sphere. As the aspect ratio gets larger and the antenna gets taller to keep the volume the same, the value of  $Q$  approaches  $\frac{3}{4}$  of that for the sphere.

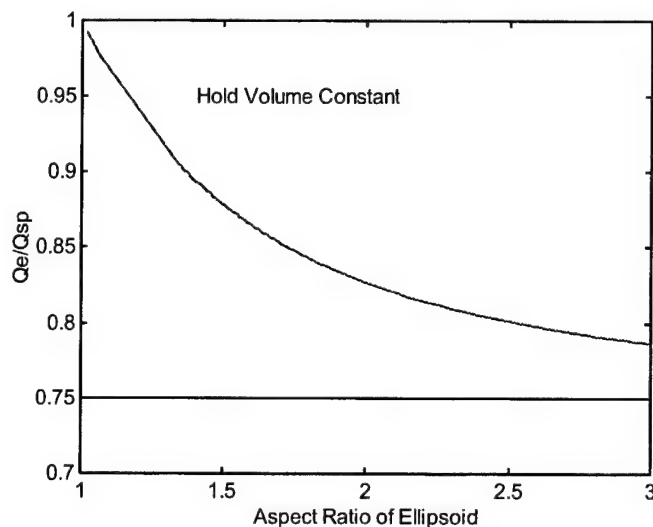


Figure 9. Ratio of  $Q$  for ellipsoid to that of sphere.

If the height of the antenna remains constant, the value of  $Q$  for the prolate spheroid is always larger than that for the sphere. This value is in accord with experience. Making the antenna thinner makes the system more resonant, and the  $Q$  becomes larger. We substitute the expression in equation (9) relating the height of the ellipsoid,  $b$ , to the distance between foci,  $d$ , in the expression for  $\beta$ , to obtain

$$Q_e = 0.75 (\lambda/2 \cdot \pi \cdot b)^3 u_m^3 [u_m/(u_m^2-1) - 0.5/u_m + 0.25(u_m^{-2}-1)\ln((u_m+1)/(u_m-1))] \quad (71)$$

We divide  $Q_e$  by Chu's expression for the sphere and set  $b = r$  to obtain

$$Q_e/Q_{\text{sphere}} = 0.75 u_m^3 [u_m/(u_m^2-1) - 0.5/u_m + 0.25(u_m^{-2}-1)\ln((u_m+1)/(u_m-1))] \quad (72)$$

Figure 10 shows the formula above as a function of the aspect ratio. As  $u_m$  becomes large, the expression in brackets approaches  $4/(3u_m^3)$ . As  $u_m$  approaches 1, the aspect ratio becomes large and the first term dominates.



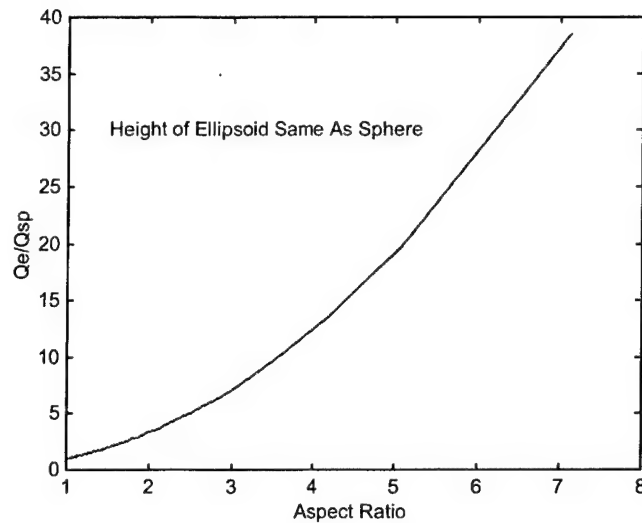


Figure 10.  $Q$  for ellipsoid compared to sphere.

In deploying an electrically small antenna, space and height will probably be constrained. For the reasons given in the introduction, the value of  $Q$  should usually be minimized. These results indicate that keeping the volume constant and allowing the height to increase leads to a 25% reduction in the value of  $Q$ . Keeping the height constant while reducing the volume leads to a large increase in  $Q$ .

## CONCLUSIONS

Treating the electromagnetic fields in prolate spheroidal coordinates permits a separation of the volume and shape-dependent effects upon the value of  $Q$ . The shape can change from spherical to cylindrical in appropriate limits. The volume can remain fixed during this transformation.

## APPENDIX: NUMERICAL METHODS FOR EVALUATING PROLATE SPHEROIDAL EIGENVALUES AND EIGENFUNCTIONS

We seek to develop a method for evaluating the eigenfunctions of the angular and radial coordinate. For the prolate spheroidal case, the equations are identical. The range of the independent variable is the primary difference between the two.

The book on numerical methods by Press et al. (1989, p. 602) suggested a relaxation method for solving for the eigenvalue, including a program listing in FORTRAN. We chose a somewhat different method, one that would provide the eigenfunction as a side benefit.

Flammer (1957) and Abramowitz and Stegun (1965) provide power series approximations to the eigenvalue that are valid for small to moderate values of  $\beta$ , the dimensionless electrical length. These approximations provide an initial input. Flammer related the eigenfunction to an infinite series of associated Legendre polynomials. A three-term recursion relation exists between the coefficients of the series. Starting with the initial approximation and the term that relates the angular function to the lowest order Legendre polynomial, we recur to ever higher order. We changed the eigenvalue slightly to evaluate the derivative of the highest order coefficient. We then changed the eigenvalue to minimize the value of the last term. (five terms would have been sufficient for four-digit accuracy. The 15 terms gave us every digit listed in Flammer's book). We then used the eigenvalue and coefficients and the normalization condition to evaluate the angular function.

The evaluation of the derivative of the angular function with respect to  $\eta$  was straightforward. We used the expression for the derivative of Legendre polynomials in terms of higher order ones. We checked the routine to an accuracy of no less than five digits by numerically differentiating the expression for the angular function.

The expression for the regular part of the radial function is given in terms of spherical Bessel functions and the same coefficients found for the angular part. Unfortunately, as expressed in other literature, the series for the irregular part in terms of spherical Neumann functions does not converge. We evaluated the integrals given on pages 53 and 54 of Flammer's book (equations (50a) and (50b) in the text) by Simpson's rule at one value of  $u$  (usually 1.077). The results for the real and imaginary parts of the function and its derivative reproduced every digit listed in Flammer's book. Only the values of the real part of the first four modes could be reproduced to six digits by the series method. We do not feel justified to use any higher modes.

Using the values of the integral and its derivative at one point as input, we solved the differential equation for the radial function by Runge Kutta methods. We compared our analytic solution and found great accuracy for almost all values of  $u$ .

We computed the integral for the energy used in calculating the value of  $Q$  by Simpson's rule. We used finite values of  $u_i$  and  $u_{i+1}$ . We then compared the values of the integral as the upper limit,  $u_i$  became large. When the value of the integral did not change in the digit, we regarded the integral as having converged to a limit for an infinite domain.

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14. ABSTRACT  This document describes the energy distribution within an electrically small antenna. Chu (1948) derived expressions for the "Q" of an omnidirectional antenna whose entire structure is enclosed within a sphere. Many authors have extended this technique. Because the sphere is not conformal to the structure of most linear polarized antennas, much energy is not accounted for. If the structure is surrounded by an ellipsoid of revolution, the fit is much better. Prolate spheroidal coordinates are expected to provide a much better description for a linearly polarized antenna. Expressions are derived for the electromagnetic field and the Poynting vector for vertically and horizontally polarized antennas. From the expressions, we calculate the Q of an electrically small antenna. An analytic expression was derived for the value of Q for the leading term. The leading term is a product of the inverse of the cube of the dimensionless wave number (product of the wavenumber and half the distance between foci of the ellipse) and a factor that depends only upon the shape. This result is reformulated so that as the shape changes, the volume and wave number remain constant.					
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